

Friction Coefficient and Berry Phase for a Topological Singularity in a Superfluid

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The coexistence of the universal transverse force and the non-universal friction force on a topological singularity in a superfluid is shown here. Based on the BCS type microscopic theory, we explicitly evaluate the quasiparticle contribution to the friction coefficient in a clean fermionic superfluid, showing a new feature of logarithmic divergence.

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I. INTRODUCTION

When a vortex in a superfluid moves, in addition to the hydrodynamic Magnus force which is proportional to the superfluid density, the coupling of the vortex to quasiparticles, phonons and impurities may cause extra forces in both transverse and longitudinal directions of the vortex motion. Those possible non-vanishing forces have been the underlying assumption for phenomenological models used in analyzing experiments¹, though analyses based on topological methods such as the Berry phase always give the universal transverse force.²

While the total transverse force on a moving vortex needs further investigations, the friction force is poorly studied. The usual quoted friction coefficient formula derived from a microscopic Hamiltonian is obtained through a relaxation time approximation in force-force correlation functions.³ However, this type of procedure is incorrect⁴, for the reason known since the 60's in the context of obtaining the friction of electrons using force-force correlation functions. The lack of an explicit and correct microscopic derivation of the friction has two consequences. First, though the friction of vortex motion is an important quantity in many experiments, it has not been received due attention. Second, the incomplete understanding of the friction has generated doubts about the exact result on the transverse force.

Here we describe a self-contained theory of both the friction and the transverse forces by a rigorous and elementary method. The method we choose follows that of Ref.[5]. In order to obtain the friction, we use a limiting process similar to the one in transport theory and in non-equilibrium statistical mechanics, which has not been explicitly discussed in Ref.[5].

In the following, after giving a general expression for the friction in Section 2, as an example, we evaluate the friction for the case of a clean fermionic superfluid in Section 3. The resulting friction coefficient is new. It comes from the off-diagonal potential scattering of the extended quasiparticles, and is stronger than the ohmic damping by a logarithmic diverging factor.

II. TRANSVERSE AND FRICTION FORCES

We consider an isolated vortex in the superfluid whose position \mathbf{r}_0 is specified by a pinning potential. The system is otherwise homogeneous and infinite. There is no externally applied supercurrent or normal current. The vortex is allowed to move slowly. Hence the system Hamiltonian H contains a slowly varying parameter $\mathbf{r}_0(t)$. The many-body wavefunction of the superfluid $|\Psi_\alpha(t)\rangle$ can be expanded in terms of the instantaneous eigenvalues $E_\alpha(\mathbf{r}_0)$ and eigenstates $|\psi_\alpha(\mathbf{r}_0)\rangle$, for which we choose phases such that $\langle\psi_\alpha|\dot{\psi}_\alpha\rangle = 0$. Because of our assumption of a homogeneous superfluid, $E_\alpha(\mathbf{r}_0)$ is independent of both \mathbf{r}_0 and time t . With those considerations, the many-body wavefunction $|\Psi_\alpha(t)\rangle$ can be expressed by

$$|\Psi_\alpha(t)\rangle = e^{-iE_\alpha t/\hbar} |\psi_\alpha(\mathbf{r}_0)\rangle + \sum_{\alpha' \neq \alpha} a_{\alpha'}(t) e^{-iE_{\alpha'} t/\hbar} |\psi_{\alpha'}(\mathbf{r}_0)\rangle, \quad (1)$$

where, to the first order in velocity, $a_\alpha(t) = 1$, and

$$a_{\alpha'}(t) = - \int_0^t dt' \langle\psi_{\alpha'}|\dot{\psi}_\alpha\rangle e^{i(E_{\alpha'} - E_\alpha)t'/\hbar}. \quad (2)$$

This gives the expectation value of the force on the vortex as

$$\begin{aligned}\mathbf{F} &= - \sum_{\alpha} f_{\alpha} \langle \Psi_{\alpha} | \nabla_0 H | \Psi_{\alpha} \rangle \\ &= - \sum_{\alpha} f_{\alpha} \langle \psi_{\alpha} | \nabla_0 H | \psi_{\alpha} \rangle + \sum_{\alpha' \neq \alpha} f_{\alpha} \langle \psi_{\alpha}(\mathbf{r}_0(t)) | \nabla_0 H | \psi_{\alpha'}(\mathbf{r}_0(t)) \rangle \times \\ &\quad \int_0^t dt' \langle \psi_{\alpha'}(\mathbf{r}_0(t')) | \dot{\psi}_{\alpha}(\mathbf{r}_0(t')) \rangle e^{i(E_{\alpha'} - E_{\alpha})(t' - t)/\hbar} + c.c. ,\end{aligned}\tag{3}$$

where f_{α} is the occupation probability of the state α . For a vortex moving with a small and uniform velocity $\mathbf{v}_V = \dot{\mathbf{r}}_0(t)$,

$$|\dot{\psi}_{\alpha}(\mathbf{r}_0)\rangle = \mathbf{v}_V \cdot |\nabla_0 \psi_{\alpha}(\mathbf{r}_0)\rangle .\tag{4}$$

Here ∇_0 denotes the partial derivative with respect to the position \mathbf{r}_0 of the pinning potential, the vortex position. The first term in the right hand side of Eq.(3) is independent of \mathbf{v}_V and will be ignored. In fact, it is zero because of the translational invariance regarding the vortex position. The integration in time can be carried out directly. In addition, we also need to use the following relations,

$$\langle \psi_{\alpha} | \nabla_0 H | \psi_{\alpha'} \rangle = (E_{\alpha'} - E_{\alpha}) \langle \psi_{\alpha} | \nabla_0 \psi_{\alpha'} \rangle = (E_{\alpha} - E_{\alpha'}) \langle \nabla_0 \psi_{\alpha} | \psi_{\alpha'} \rangle ,\tag{5}$$

which are obtained by taking gradient ∇_0 with respect to $H|\psi_{\alpha'}\rangle = E_{\alpha'}|\psi_{\alpha'}\rangle$ and $\langle \psi_{\alpha} | H = E_{\alpha} \langle \psi_{\alpha} |$, then multiplying from left or right by $\langle \psi_{\alpha} |$ or $|\psi_{\alpha'}\rangle$ respectively.

To obtain the long time behavior of Eq.(3) requires a limiting procedure. There are two ways of taking limiting sequences. A possible sequence is to take the low frequency limit before the thermodynamic limit. As we will find out, the transverse force is independent of the limiting process. Therefore such a limiting sequence is correct as long as only the transverse force is concerned. In such a calculation, because the energy levels have been treated as discrete ones, there is no friction. In order to obtain friction, we have to take the thermodynamic limit before the low frequency limit. For this purpose, we use the Laplace average⁶, $\lim_{t \rightarrow \infty} \mathbf{F}(t) = \lim_{\epsilon \rightarrow 0^+} \mathbf{F}(\epsilon)$, where $\mathbf{F}(\epsilon) = \epsilon \int_0^{\infty} dt \mathbf{F}(t) e^{-\epsilon t}$, and $\mathbf{F}(\epsilon = 0) = \mathbf{F}_{\perp} + \mathbf{F}_{\parallel}$. We will also use the identity $1/(\epsilon + ix) = \pi \delta(x) - iP(1/x)$. For the transverse force \mathbf{F}_{\perp} , we have

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \int_0^{\infty} dt e^{-\epsilon t} (1 - \cos(E_{\alpha} - E_{\alpha'})t/\hbar) = \lim_{\epsilon \rightarrow 0^+} (1 - \pi \delta(E_{\alpha} - E_{\alpha'})\epsilon) = 1 ,$$

because the summation over states in Eq.(3) is well behaved regardless whether E_{α} is discrete or continuous. Hence

$$\mathbf{F}_{\perp} = i\hbar \sum_{\alpha' \neq \alpha} f_{\alpha} \{ (\langle \psi_{\alpha} | \nabla_0 \psi_{\alpha'} \rangle \times \langle \nabla_0 \psi_{\alpha'} | \psi_{\alpha} \rangle) \cdot \hat{\mathbf{z}} \} \mathbf{v}_V \times \hat{\mathbf{z}} .\tag{6}$$

This is precisely what has been obtained in Ref.[5], where further calculations lead to $\mathbf{F}_{\perp} = -\hbar L \rho_s \mathbf{v}_V \times \hat{\mathbf{z}}$, independent of details of the system. Here ρ_s is the superfluid number density, L the length of the vortex. The universal nature of the transverse force has been confirmed in experiments of the vibrating wire⁷ and the vortex precession⁸, and even in dirty superconductors⁹. The longitudinal force, friction, is given by $\mathbf{F}_{\parallel} = -\eta \mathbf{v}_V$ with

$$\eta = \frac{\pi}{2} \sum_{\alpha' \neq \alpha} \hbar \frac{f_{\alpha} - f_{\alpha'}}{E_{\alpha'} - E_{\alpha}} \delta(E_{\alpha} - E_{\alpha'}) |\langle \psi_{\alpha} | \nabla_0 H | \psi_{\alpha'} \rangle|^2 .\tag{7}$$

The friction coefficient η is determined by low energy excitations such as phonons, extended quasiparticles, and localized quasiparticles when their discrete energy spectrum is smeared out by impurities. This expression is identical to the result in Ref.[10] for the case of ohmic damping in the zero frequency limit. Eq.(7) will not pick up any superohmic contributions, and will give infinity for any subohmic contributions. We point out that with the aid of Eq.(5) the transverse force can be expressed only in terms of the wavefunction or the density matrix without explicit referring to the Hamiltonian or its eigenvalues, as shown by Eq.(6), on the other hand, the longitudinal force, the friction, cannot, as shown by Eq.(7). The explicit dependence on the Hamiltonian or its eigenvalues in Eq.(7) is the source of the sensitivity of friction to details of the system.

III. QUASIPARTICLE CONTRIBUTION

For the superfluid ^4He , there is no full microscopic theory yet¹. We will not attempt to evaluate Eq.(7) for this superfluid here because of the sensitivity of the friction to details. The situation can be very different in the case of the recent Bose-Einstein condensed systems, where a well defined microscopic theory is supposed to be known. Instead, as an example to illustrate the directness and usefulness, we evaluate Eq.(7) for the case of a homogeneous fermionic superfluid using BCS theory with s-wave pairing. At finite temperatures the extended states above the Fermi level, the quasiparticles, are partially occupied. The vortex motion causes transitions between these states and gives rise to friction. The transitions between different single quasiparticle levels $\langle\psi_\alpha|\nabla_0 H|\psi_{\alpha'}\rangle$ are considered here since they dominate the low energy process. The quasiparticles are described by the eigenstates, u_α and v_α , of the Bogoliubov-de Gennes equation. Their behavior in the presence of a vortex has been well studied in Ref.[11]. We may take

$$|\psi_\alpha\rangle = \begin{pmatrix} u_\alpha(x) \\ v_\alpha(x) \end{pmatrix} = \frac{1}{\sqrt{L}} e^{ik_z z} e^{i\mu\theta + i\sigma_z\theta/2} \hat{f}(r), \quad (8)$$

with \mathbf{r} measured from the vortex position, and θ the azimuthal angle around the vortex. In order to obtain an analytical form for the transition element, we use a WKB solution for $\hat{f}(r)$,

$$\hat{f}(r) = \frac{1}{2\sqrt{R}r} \begin{pmatrix} [1 \pm \sqrt{E^2 - |\Delta(r)|^2}/E]^{1/2} \\ [1 \mp \sqrt{E^2 - |\Delta(r)|^2}/E]^{1/2} \end{pmatrix} \times \\ \exp \left\{ i \int_{r_t}^r dr' \left(k_\rho^2 \frac{r'^2 - r_t^2}{r'^2} \pm \frac{2m}{\hbar^2} \sqrt{E^2 - |\Delta(r')|^2} \right) \right\} + c.c. . \quad (9)$$

Here $k_\rho^2 = k_f^2 - k_z^2$, R is the radial size of the system. This WKB solution is valid when r is outside the classical turning point $r_t = |\mu|/k_\rho$. Here r_t is the impact parameter. A WKB solution also exists inside the turning point. However, because it approaches zero as $(rk_\rho)^{|\mu|}/|\mu|!$, the contribution to the transition elements from this region is small, and will be set to zero. The transition elements are then given by

$$\begin{aligned} |\langle\psi_\alpha|\nabla_0 H|\psi_{\alpha'}\rangle|^2 &= \left| \int dx (u_{\alpha'}^*(x) \nabla_0 \Delta v_\alpha(x) + v_{\alpha'}^*(x) \nabla_0 \Delta^* u_\alpha(x)) \right|^2 \\ &= \begin{cases} \frac{\Delta_\infty^4}{E^2 R^2} \delta_{k_{z1}, k_{z2}} \delta_{\mu_1, \mu_2 \pm 1}, & |\mu| \leq \xi_0 k_\rho \\ 0, & |\mu| > \xi_0 k_\rho \end{cases} . \end{aligned} \quad (10)$$

Here Δ_∞ is the value of $|\Delta(r)|$ far away from the vortex core. Physically it means that if the classical quasiparticle trajectory is far away from the vortex core, it will not contribute to the friction. The summation over states in Eq.(7) is replaced by

$$\sum_{\alpha' \neq \alpha} = \sum_{\mu_1, \mu_2, k_{z1}, k_{z2}} \int dE_1 dE_2 \frac{E_1}{\sqrt{E_1^2 - \Delta_\infty^2}} \frac{E_2}{\sqrt{E_2^2 - \Delta_\infty^2}} \left(\frac{m}{\hbar^2 k_f} \right)^2 \frac{R^2}{\pi^2},$$

after considering the density of states.

Substituting Eq.(10) into Eq.(7), using the quasiparticle distribution function $f_\alpha = 1/(e^{\beta E_\alpha} + 1)$, the coefficient of friction is given by

$$\eta = \frac{Lm^2 \xi_0 \Delta_\infty^4 \beta}{8\pi^2 \hbar^3} \int_{\Delta_0}^{\infty} dE \frac{E^2}{E^2 - \Delta_\infty^2} \frac{1}{E^2 \cosh^2(\beta E/2)}. \quad (11)$$

The integral in Eq.(11) diverges logarithmically. It implies that the spectral function corresponding to the vortex-quasiparticle coupling is not strictly ohmic but has an extra frequency factor proportional to $\ln(\Delta_\infty/\hbar\omega)^{12,10}$. When $\hbar\omega$ is not very small comparing to Δ_∞ , which may be realized when close to T_c , we can ignore the logarithmic divergence in Eq.(11) by using the density of states for normal electrons to obtain a finite friction, i.e. replacing $E^2/(E^2 - \Delta_\infty^2)$ with 1 in Eq.(11). Close to T_c , the friction approaches zero the same way as Δ_∞^2 , which is proportional to the superfluid density ρ_s . When $-\ln(\hbar\omega/\Delta_\infty)$ is large, we need to use a more accurate expression of vortex friction obtained in Ref.[10]. Straightforward evaluation shows that in such a case

$$\eta = \frac{Lm^2 \xi_0 \Delta_\infty^3 \beta}{16\pi^2 \hbar^3} \frac{1}{\cosh^2(\beta \Delta_\infty/2)} \ln(\Delta_\infty/\hbar\omega_c). \quad (12)$$

Here ω_c is the low frequency cut-off. It is determined by the size of the system for a single vortex, and by the inter-vortex distance for a vortex array.

It should be emphasized that the logarithmic divergence comes from the interplay between the divergence in the density of states and the off-diagonal potential scattering. We can consider a situation in which we physically create a pinning center to trap the vortex and guide its motion. In such a case the vortex has a diagonal potential. If the scattering is dominated by the diagonal potential, e.g., by the pinning potential, an additional factor coming from $|u_\alpha|^2 - |v_\alpha|^2$ will remove this logarithmic divergence. This again shows the sensitivity of the friction to system details.

The above results apply to the dynamics of the axisymmetric vortex in ^3He B phase, where the Bogoliubov-de Gennes equation is essentially the same as the one in a s-wave superconductor.

IV. CONCLUSION

The transverse force formula confirms the Berry phase results obtained with a trial many-body wavefunction. New results on the friction have been obtained from quasiparticle contributions, which can be further tested experimentally.

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